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Thermal instability with radiation by the method of energy

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Abstract—Energy-stability theory is applied to the case of radiation heat transfer in an optically thin, quiescent fluid layer heated from below and bounded by rigid, black, perfectly conducting planes. The radiation term in the energy equation destroys the quadratic character of the energy identity. It is shown, however, that the right-hand side of the energy identity can be bounded by a suitable quadratic term for all physically allowable disturbances. The result is a conditional stability limit dependent upon disturbance amplitude. Results are computed for a variety of cases; these are compared to existing linear and energy stability results.

1. INTRODUCTION

Energy-stability theory dates to the early work of Reynolds [1] and Orr [2], but owes its modern incarnation to Serrin [3] and Joseph [4]. It provides a sufficient condition for stability, *usually* to disturbances of arbitrary amplitude, which is complementary to the sufficient condition for instability to infinitesimal disturbances given by linear-stability theory. For certain instability mechanisms, the results of energy and linear theories are quite close, restricting the range of potential subcritical instability for the basic state under consideration. In the case of the onset of convection in a quiescent, nonradiating Boussinesq fluid bounded by rigid, perfectly conducting planes, the results of the theories are identical, resulting in a global stability limit.

Energy theory has recently shown promise in providing stability boundaries for the instability of thermocapillary convection in a model of the float-zone crystal-growth problem [5, 6]. In this problem, however, a simple convective heat-transfer mechanism was assumed to hold at the free surface, while the actual crystal-growth application is obviously influenced strongly by radiative heat transfer. The purpose of this research is to consider the influence of radiation on an energy-stability analysis of the thermal instability problem. The difficulty encountered in incorporating radiative effects is associated with the nonlinearity of the temperature in the energy equation. This nonlinearity results in an energy identity which contains a nonquadratic functional. For some types of nonquadratic functionals, it is possible [7, 8] to use a spatially weighted energy to overcome the restriction of a *conditional* stability limit, i.e. one which is valid for disturbances less than a certain amplitude. We shall show for the problem considered here that, for physically allowable disturbances, all nonquadratic terms in this functional are stabilizing and may be bounded by an appropriate quadratic term, reducing the determination of stability limits to the solution of a standard eigenvalue problem. A conditional stability limit is obtained and the amount of restriction imposed by this condition will be seen to depend, not surprisingly, on the strength of the radiative heat transfer.

The following sections describe the governing equations, implementation of energy theory and the results obtained. Comparisons with previous work are possible in a couple of limiting cases, namely: (i) in the absence of radiation, where the results should be in agreement with those from classical analysis of thermal instability; and (ii) restriction to small temperature differences and infinitesimally small disturbances, in which case energy and linear theories coincide and the linear stability results of Christophorides and Davis [9] are available.

2. GOVERNING EQUATIONS AND BASIC STATE

Consider an optically thin, Boussinesq fluid contained between a pair of black, perfectly conducting, horizontal planes a distance d apart as illustrated in Fig. 1. The lower plane at $z = 0$ is held at the constant temperature T_H and the upper at constant temperature $T_C < T_H$. The momentum and continuity equations, subject to the Boussinesq approximation, are given by

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho_C}\right) + \nu\nabla^2\mathbf{v} + \frac{\rho}{\rho_C}g\mathbf{k} \qquad (1)$$

and

$$\nabla\cdot\mathbf{v} = 0. \qquad (2)$$

The energy balance includes radiation heat transfer and incorporates the assumptions of an optically thin

NOMENCLATURE

a	horizontal wavenumber
B	disturbance bound
c_p	specific heat
d	layer thickness
D	dissipation functional
E	energy functional
$\mathbf{g}$	gravitational acceleration
H	space of kinematically admissible functions
I	production functional
k	thermal conductivity
n	fluid index of refraction
N_r	radiation parameter
p	pressure
P	basic-state pressure
Pr	Prandtl number
Ra	Rayleigh number
t	time
T	temperature
$\mathbf{v} = (u, v, w)$	velocity
V	integration volume
$\mathbf{x} = (x, y, z)$	spatial coordinates.

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
ΔT	temperature difference
η	overheat parameter
Θ	basic-state temperature
θ	disturbance temperature
κ	fluid absorption coefficient
λ	coupling parameter
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density
σ	Stefan–Boltzmann constant
ϕ	modified disturbance temperature
Ψ	radiation function.

Subscripts

C	cold-wall or core value
E	energy limit
H	hot-wall value
x, y	spatial direction.

fluid and either (i) an absorption coefficient that is independent of temperature, pressure and wavelength, or (ii) a Planck mean absorption coefficient that is independent of temperature and pressure. The resulting equation is

$$T_t + \mathbf{v}\cdot\nabla T = \alpha\nabla^2 T - \frac{4\kappa n^2\sigma}{\rho_C c_p}\left[T^4 - \frac{1}{2}(T_C^4 + T_H^4)\right]. \tag{3}$$

In the above equations, $\mathbf{v}$ is the velocity vector, p the pressure and T the temperature. The quantity ρ_C is the cold-wall fluid density from the equation of state

$$\rho = \rho_C[1 - \beta(T - T_C)]. \tag{4}$$

Other fluid properties (assumed constant throughout the layer) are β the coefficient of volumetric expansion, α the thermal diffusivity, $\nu = \mu/\rho_C$ the kinematic viscosity, c_p the specific heat at constant pressure, κ the absorption coefficient and n the index of refraction. Finally, $\mathbf{g}$ is the gravitational acceleration and σ is the Stefan–Boltzmann constant. The boundary conditions which accompany equations (1)–(3) under the assumptions stated above are:

$$\text{at } z = 0, \quad \mathbf{v} = \mathbf{0}, T = T_H \tag{5}$$

$$\text{and at } z = d, \quad \mathbf{v} = \mathbf{0}, T = T_C. \tag{6}$$

Note that the use of a Planck mean absorption coefficient that is *independent of temperature* implies a specific dependence between the wavelength and the temperature in the absorption coefficient. For example, if $\kappa_\lambda(\lambda, T)$, one possibility would be to require $\kappa_\lambda(1/T, T) = \text{constant}$. Thus, the use of the Planck mean absorption coefficient may not be as general as it seems.

We seek a motionless basic state of the form $\mathbf{v} = \mathbf{0}$ and $(p, T) = (P(z), \Theta(z))$, satisfying equations (1)–(3), subject to equations (5) and (6). Introducing the length scale d and a dimensionless basic-state temperature $\Theta^* = (\Theta - T_C)/\Delta T$, where $\Delta T = T_H - T_C$, the basic-state temperature is governed by (dropping asterisks)

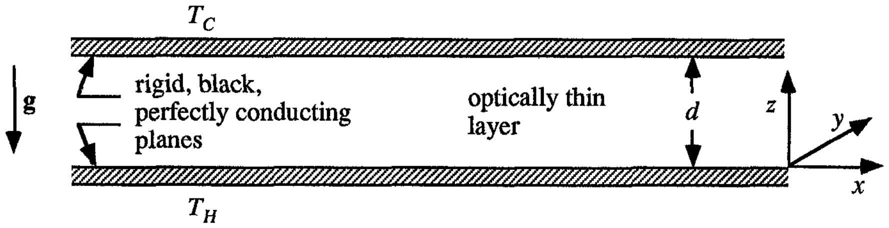

Fig. 1. A schematic of the optically thin liquid layer heated from below.

$$\Theta'' - \frac{N_r \eta^3}{4}\left[\left(\Theta + \frac{1}{\eta}\right)^4 - \frac{1}{2}\left(1 + \frac{1}{\eta}\right)^4 - \frac{1}{2\eta^4}\right] = 0, \quad (7)$$

with

$$\Theta(0) = 1 \quad (8)$$

and

$$\Theta(1) = 0. \quad (9)$$

Two dimensionless groups appear in equation (7): the *radiation–conduction parameter* N_r, defined as

$$N_r = \frac{16\kappa n^2 \sigma d^2 T_C^3}{k}, \quad (10)$$

which is the ratio of radiative to conductive heat transfer; and an *overheat parameter* η, defined as

$$\eta = \Delta T / T_C. \quad (11)$$

In equation (10), k is the thermal conductivity. Solutions to equations (7)–(9) were determined numerically using a standard shooting technique. Further discussion of the motionless basic-state problem without the optically-thin approximation can be found in the standard texts by Siegel and Howell [10] and Özisik [11].

Basic-state temperature profiles are shown in Fig. 2 for various values of the radiation and overheat parameters. Figure 2(a) shows profiles for a fixed overheat and various levels of radiation. In the limit of negligible radiation heat transfer ($N_r \to 0$), one recovers the usual linear temperature variation between the two plates. As radiation becomes more important, one observes the development of thermal boundary layers at both plates with the central core approaching a uniform temperature. That this will occur for large N_r is clear from the form of equation (7), since large N_r is equivalent to having a small parameter multiplying the most highly differentiated term. A matched asymptotic analysis of this problem yields, at leading order, a core temperature Θ_C given by

$$\Theta_C = -\frac{1}{\eta} + \left[\frac{1}{2}\left(1 + \frac{1}{\eta}\right)^4 + \frac{1}{2\eta^4}\right]^{1/4}. \quad (12)$$

For $\eta = 0.3$, equation (12) yields a value of $\Theta_C = 0.59455$, in excellent agreement with the numerical result of Fig. 2(a) for $N_r = 100$, in which $\Theta(1/2) = 0.5942$. As η approaches zero, the core temperature becomes 1/2—the mean temperature of the two plates. This is reasonable because, in this limit, the core fluid absorbs radiation equally from both boundary plates. This equality of absorption occurs when either $\Delta T \to 0$ and so $T_H \to T_C$, or when ΔT stays fixed and both T_H and $T_C \to \infty$. Thus, under the assumption of black walls and an optically thin layer, the temperature structure observed with increasing radiation–conduction parameter is in accord with intuition.

The effect of varying the overheat parameter is

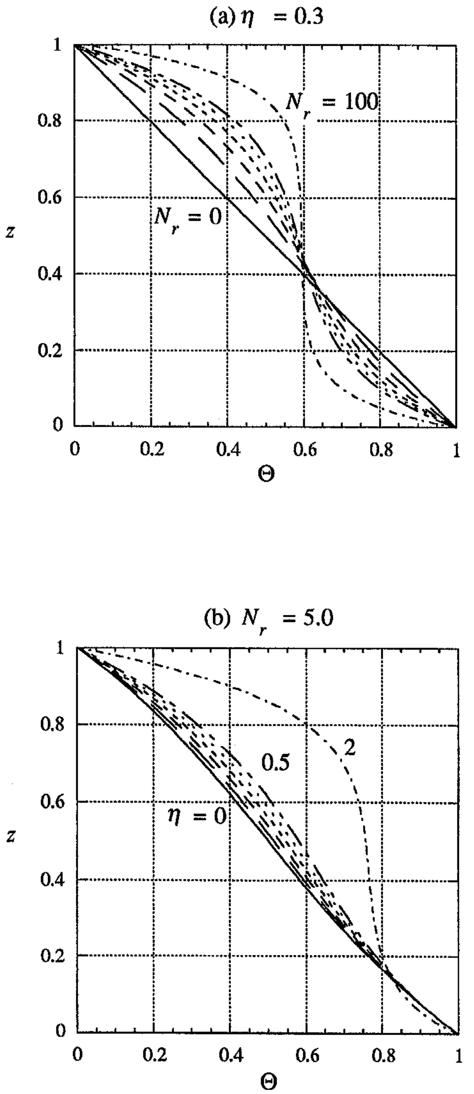

Fig. 2. The basic-state temperature profile for: (a) $\eta = 0.3$ and $N_r = 0, 5, 10, 15, 20, 25$ and 100; and (b) $N_r = 5.0$ and $\eta = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ and 2.0.

shown in Fig. 2(b). For the moderate value of N_r shown, the effect of increasing overheat is to skew the core temperature toward that of the hot wall. This again is in keeping with intuition.

3. ENERGY-STABILITY ANALYSIS

The analysis begins in the standard fashion by assuming a solution of the form

$$[\mathbf{v}(\mathbf{x}, t), p(\mathbf{x}, t), T(\mathbf{x}, t)] = [\mathbf{0}, P(z), \Theta(z)] + [\mathbf{v}'(\mathbf{x}, t), p'(\mathbf{x}, t), \theta(\mathbf{x}, t)], \quad (13)$$

where $P(z)$ is the hydrostatic basic-state pressure field and the final bracketed quantity represents disturbances to the basic state. Substituting this form into governing equations (1)–(3) and using scales of α/d, ΔT, $\mu\alpha/d^2$ and d^2/α for the disturbance velocity, temperature, pressure and time, respectively, and using a reference temperature of T_C, one arrives at the *disturbance equations* (dropping primes):

$$Pr^{-1}\{\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}\} = -\nabla p+\nabla^2\mathbf{v}+\sqrt{Ra}\,\phi\mathbf{k} \quad (14)$$

$$\phi_t+\mathbf{v}\cdot\nabla\phi+\sqrt{Ra}\,\mathbf{v}\cdot\nabla\Theta = \nabla^2\phi-\frac{N_r\eta^3}{4}\Psi \quad (15)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (16)$$

where we have replaced θ by $\phi = \theta\sqrt{Ra}$, and

$$\Psi = \frac{\phi^4}{\sqrt{Ra^3}}+\frac{4\phi^3}{Ra}\left(\Theta+\frac{T_C}{\Delta T}\right)+\frac{6\phi^2}{\sqrt{Ra}}\left(\Theta+\frac{T_C}{\Delta T}\right)^2 +4\phi\left(\Theta+\frac{T_C}{\Delta T}\right)^3. \quad (17)$$

Two additional dimensionless parameters appear in these equations, the *Prandtl number* $Pr = \nu/\alpha$ and the *Rayleigh number*

$$Ra = \frac{g\beta\Delta T d^3}{\alpha\nu}.$$

Energy-stability theory is applied at this point by taking the inner product of the disturbance velocity $\mathbf{v} = (u, v, w)$ with equation (14), multiplying the disturbance energy equation (15) by ϕ, coupling (using a new parameter $\lambda > 0$) and integrating the result over a volume V, obtaining the *energy identity*

$$\frac{dE}{dt} = -D+I\sqrt{Ra}-\lambda\frac{N_r\eta^3}{4}\int_V \phi\Psi\,dV, \quad (18)$$

where

$$E = \frac{1}{2}\int_V \{Pr^{-1}|\mathbf{v}|^2+\lambda\phi^2\}\,dV \quad (energy) \quad (19)$$

$$D = \int_V \{|\nabla\mathbf{v}|^2+\lambda|\nabla\phi|^2\}\,dV \quad (dissipation) \quad (20)$$

and

$$I = \int_V w\phi[1-\lambda\Theta']\,dV \quad (production), \quad (21)$$

where a prime on Θ denotes differentiation with respect to z.

The volume V is chosen assuming disturbances which are periodic in the x- and y-directions with wavenumbers a_x and a_y, respectively, and is the cell

$$V = \left\{(x,y,z)\,\middle|\,0 \leqslant x \leqslant \frac{2\pi}{a_x},\ 0 \leqslant y \leqslant \frac{2\pi}{a_y},\ 0 \leqslant z \leqslant 1\right\}.$$

The *coupling parameter* λ is a parameter whose value is arbitrary, since it serves merely to link together the disturbance kinetic and thermal "energies" to form the energy functional E. Since it is desired to determine the *largest* value of the Rayleigh number (the chosen stability parameter) below which stability is guaranteed, λ is selected, through numerical experimentation, to optimize this result [12].

When radiation is negligible ($N_r = 0$), the last functional in equation (18) is absent and the remaining functionals in the energy identity are quadratic in disturbance quantities. The maximum problem obtained by bounding the right-hand side of what remains in equation (18) leads to a set of Euler–Lagrange equations that are identical to the linear stability equations, yielding an energy limit equal to the linear limit of $Ra = 1708$.

For the case of interest here, $N_r \neq 0$ and the effect of radiation is always stabilizing. In order to demonstrate this, it is necessary to show that the quantity $\phi\Psi$ is nonnegative-definite for all physically allowable disturbances. First we define $\zeta \equiv \bar{\Theta}/\Delta T = \Theta+T_C/\Delta T$, where $\bar{\Theta}$ is the *dimensional* basic-state absolute temperature. Note that $T_C \leqslant \bar{\Theta} \leqslant T_H$ and so $1/\eta \leqslant \zeta \leqslant 1+1/\eta$. Next, we define

$$x = \phi+\zeta\sqrt{Ra} = \frac{\sqrt{Ra}}{\Delta T}(\overline{\theta'}+\bar{\Theta}) = \frac{\sqrt{Ra}}{\Delta T}\bar{T},$$

where $\overline{\theta'}$ is the *dimensional* disturbance temperature and $\bar{T}$ is the *dimensional* absolute temperature. Since $\bar{T} \geqslant 0$, we have $x \geqslant 0$ for all physically allowable temperatures. Now, we write the quantity $\phi\Psi$ in terms of ζ and x as

$$\phi\Psi = \frac{\phi^2}{\sqrt{Ra^3}}[x^3+\zeta\sqrt{Ra}\,x^2+\zeta^2 Ra\,x+\zeta^3\sqrt{Ra^3}]. \quad (22)$$

Since $x \geqslant 0$ and $\zeta > 0$, we see that $\phi\Psi$ is nonnegative-definite. Thus, radiation heat transfer is proved to be stabilizing in the situation considered here.

In order to proceed further with the energy-theory analysis, the quintic character of the functional in equation (18) involving the terms in equation (22) needs to be examined. From the fact that $x \geqslant 0$, we can write the greatest lower bound on ϕ as

$$\phi \geqslant -\sqrt{Ra}\min\zeta = -\sqrt{Ra}/\eta. \quad (23)$$

This suggests the following definition of a *bounding parameter* B:

$$\phi \geqslant -B\sqrt{Ra}/\eta, \quad 0 \leqslant B \leqslant 1. \quad (24)$$

This implies that

$$x \geqslant (1-B)\sqrt{Ra}/\eta$$

and, from equation (22), we find

$$\phi\Psi \geqslant M\phi^2/\eta^3, \quad (25)$$

where

$$M = (1-B)^3+(1-B)^2+2-B$$

and

$$1 \leqslant M \leqslant 4.$$

Applying inequality (25) to the energy identity (18) results in the inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant -D + I\sqrt{Ra} - \lambda \frac{N_r M}{4} \int_V \phi^2 \,\mathrm{d}V, \quad (26)$$

in which all terms are now of standard quadratic form in disturbance quantities. Finally, we use the inequality (24) to write a proper bound on the *amplitude* of the thermal disturbance in the form

$$|\phi| \leqslant B\sqrt{Ra}/\eta, \quad 0 \leqslant B \leqslant 1. \quad (27)$$

If a disturbance satisfies the bound (27), then it also satisfies the bound (24) and the energy inequality (26) follows. The result of this bound (27) on the amplitude of the thermal disturbance is that the stability limit determined from the modified energy identity (26) is a *conditional stability limit* that is dependent upon the disturbance amplitude through the parameter B. Note that, if $B = 1$, we can use the less restrictive bound (24) and say that the conditional stability limit is valid for all physically allowable thermal disturbances.

We now apply the reformulated energy theory of Davis and von Kerczek [13] and further bound the right-hand side of equation (26), viz.

$$\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant \nu \equiv \max_H \left\{ \frac{-D + I\sqrt{Ra} - \lambda \frac{N_r M}{4} \int_V \phi^2 \,\mathrm{d}V}{E} \right\}, \quad (28)$$

where the maximum is taken over the space H of kinematically admissible functions,

$$H = \{\mathbf{v}, \phi \,|\, \mathbf{v} = \phi = 0 \quad \text{on} \quad z = 0, 1; \nabla\cdot\mathbf{v} = 0\}.$$

Asymptotic stability in the mean (i.e. $E \to 0$ as $t \to \infty$) is guaranteed for $\nu < 0$. Hence, for fixed coupling parameter λ, the energy limit corresponds to the largest value of Ra for which $\nu = 0$. From equation (28), one may derive an equivalent set of Euler–Lagrange equations; these may be further manipulated in standard fashion to eliminate all variables in favor of w and ϕ. Further defining the quantities W and Φ by

$$w(x, y, z) = W(z)H(x, y)$$

and

$$\phi(x, y, z) = \Phi(z)H(x, y),$$

where

$$\nabla_h^2 H + a^2 H = 0,$$

$$a = \sqrt{a_x^2 + a_y^2}$$

is the horizontal wavenumber and $\nabla_h^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the horizontal Laplacian, allows the problem for the determination of the energy limit to be reduced to the solution of the sixth-order eigenvalue problem:

$$(D^2 - a^2)^2 W - \frac{1}{2} a^2 \sqrt{Ra}\,(1 - \lambda\Theta')\Phi = 0 \quad (29)$$

$$(D^2 - a^2)\Phi + \frac{\sqrt{Ra}}{2\lambda}(1 - \lambda\Theta')W - \frac{1}{4} N_r M \Phi = 0, \quad (30)$$

subject to the conditions

$$W = DW = \Phi = 0 \quad \text{on} \quad z = 0, 1. \quad (31)$$

In the above, D and a prime both indicate differentiation with respect to z.

The energy stability limit Ra_E is now determined as

$$Ra_E = \max_{\lambda > 0} \min_a Ra^*, \quad (32)$$

where Ra^* is the smallest positive eigenvalue of system (29)–(31). For given values of N_r and η, the basic state was determined, as outlined in the previous section. The linear eigenvalue problem was solved using standard shooting and the numerical algorithm was checked against the published results of Joseph and Shir [12]. Values of Ra^* were determined for fixed values of parameters N_r, a, λ and the bounding parameter B. Subsequent variation of a and λ permits the determination of Ra_E according to the procedure of equation (32). For values of the Rayleigh number $Ra < Ra_E$, the quiescent state is stable to disturbances obeying the bound given by equation (27).

4. RESULTS AND DISCUSSION

The calculations just described were performed on an IBM RISC System/6000 computer for a variety of parameters to assess the influence of radiation heat transfer and disturbance bounds on the stability boundary. Recall that the bounding parameter B was restricted to lie between zero and unity, with zero representing a negligible disturbance amplitude and unity all physically allowable thermal disturbances. It can further be shown that, for $B = 0$, the Euler–Lagrange system is equivalent to that governing the linear stability problem. Christophorides and Davis [9] did a linear stability analysis under the assumption of small ΔT, corresponding to $\eta \to 0$ in our notation. We computed the cases $(N_r, B, \eta) = (1, 0, 0)$ and $(N_r, B, \eta) = (5, 0, 0)$ and agreed with their linear stability results to about 0.01%.

Figure 3 shows the influence of the radiation and bounding parameters for a fixed value of the overheat parameter ($\eta = 0.3$). The lowest, solid curve on the figure corresponds to the absence of radiation heat transfer ($N_r = 0$) and is at the value $Ra_E = 1708$, which is the classical stability boundary for rigid, perfectly conducting walls [12]. Since $N_r = 0$, the influence of the bounding parameter disappears from the result, as is easily observed from equation (26), consistent with the coincidence of the energy and linear stability limits of the classical problem.

The influence of the bounding parameter B is, not

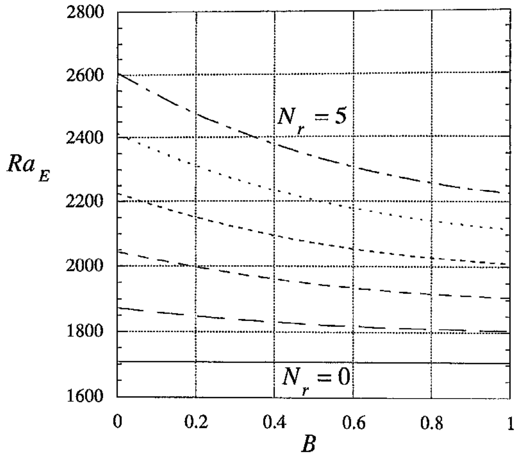

Fig. 3. The energy stability limit Ra_E vs the bounding parameter B for $\eta = 0.3$ and $N_r = 0, 1, 2, 3, 4$ and 5.

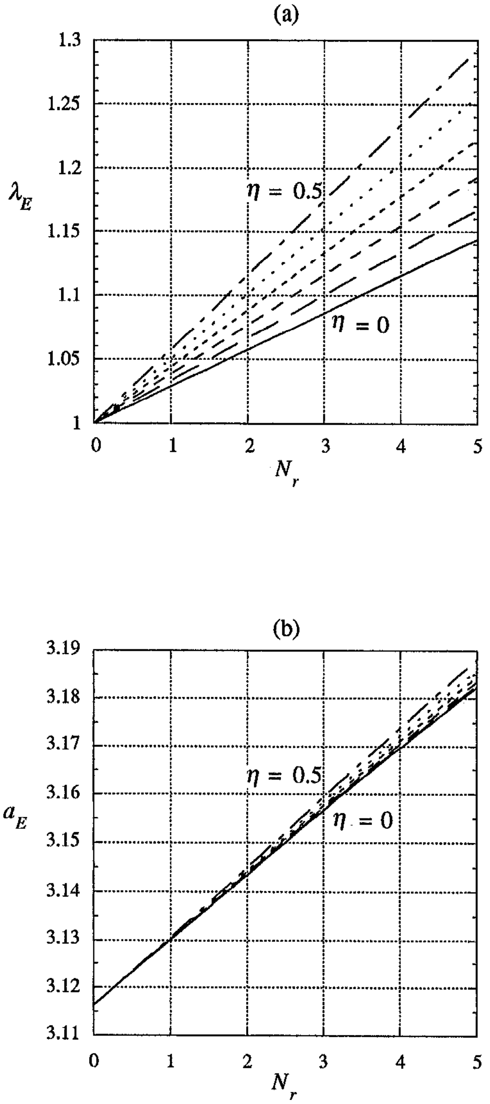

Fig. 5. The minimizing value of (a) the coupling parameter λ_E and (b) the horizontal wavenumber a_E vs the radiation–conduction parameter N_r for all physically allowable disturbances ($B = 1$) and $\eta = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5.

surprisingly, to increase the energy stability limit for $B \to 0$. For a fixed value of N_r, the magnitude of Ra_E increases as B decreases. The intersections of the curves for different values of the radiation parameter with the vertical axis $B = 0$ provide the linear stability limits for these values of N_r and $\eta = 0.3$.

The stabilization provided by including radiation heat transfer is also partially illustrated by Fig. 3. For a fixed value of the bounding parameter, increasing N_r results in a larger value of Ra_E. It was demonstrated analytically that this would occur by showing that the energy-identity functional corresponding to radiative transfer was nonnegative-definite. As pointed out by Christophorides and Davis [9], this stabilization may also be anticipated on physical grounds from a re-examination of the basic-state temperature profiles of Fig. 2. The effect of increasing both N_r and η is to cause the temperature variations within the layer to become confined (for asymptotically large values of both parameters) to boundary layer regions near both walls. The net effect of this layer formation is to decrease the effective length scales over which the temperature variations occur, therefore reducing the *effective* Rayleigh number. Figure 4 shows the effect of the overheat parameter η for all physically allowable disturbances ($B = 1$). Again, the effect of increasing η is to provide stabilization, but *only* through its influence on the basic state, since η, unlike N_r, does not appear explicitly on the right-hand side of equation (28).

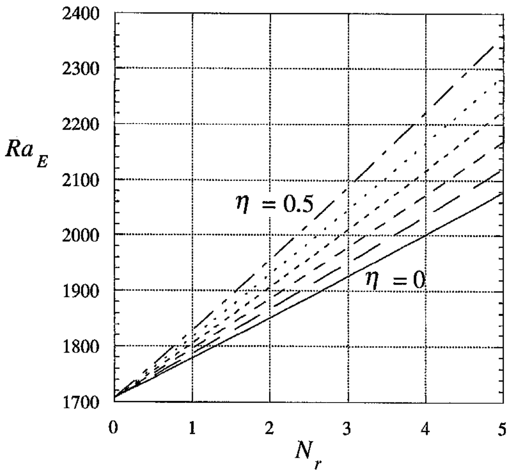

Fig. 4. The energy stability limit Ra_E vs the radiation–conduction parameter N_r for all physically allowable disturbances ($B = 1$) and $\eta = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5.

Recall that the determination of Ra_E requires that the results of a linear eigenvalue calculation be maximized over the coupling parameter λ and minimized over the horizontal wavenumber a according to equation (32). Computations of energy stability limits for thermocapillary convection in half-zone models of the float-zone problem [5, 6] found that changes in λ could bring about order-of-magnitude variations in the energy stability limit and that the maximizing value of this parameter was extremely small [$O(10^{-7})$], indicating the lack of importance of thermal disturbances in determining the stability limit. For this problem, like the standard problem without radiation, this is not the case as observed in Fig. 5(a), which shows the variation of λ_E, the maximizing value of λ, with N_r and η for all physically allowable disturbances. The increase of λ_E with increasing radiation may be loosely interpreted as representing the increased importance of thermal disturbances in the energy functional. The corresponding variation of the minimizing horizontal wavenumber a_E is given in Fig. 5(b). Its increase with increasing radiation is physically consistent with the decrease in the size of the effective length scales discussed above; however, the variation is slight.

5. SUMMARY

Energy-stability theory was applied to a basic state of a quiescent, optically thin, horizontal fluid layer heated from below, bounded by two black, isothermal walls and subject to the effects of radiation heat transfer. For physically allowable disturbances, it is possible to bound the nonquadratic functional in the energy identity by an appropriate quadratic term, resulting in the calculation of an energy limit conditional upon disturbance amplitude. Radiation was found to be stabilizing in all cases, with the amount of stabilization increasing with increasing radiation and overheat parameters and with decreasing disturbance amplitude. The stabilization is physically explained, in part, by the deformation of the usual (in the absence of radiation) linear temperature profile between the two planes, resulting in the formation of thermal boundary layers at both walls. The calculated results are in agreement, in the appropriate limiting cases, with energy theory results without radiation and with linear theory results valid for small values of the overheat parameter.

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